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Author: Michel Talagrand
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Maharam's problem

Michel Talagrand ¹
 Université Paris VI and the Ohio State University
 Dedicated to J. W. Roberts

Abstract

We construct an exhaustive submeasure that is not equivalent to a measure. This solves problems of J. von Neumann (1937) and D. Maharam (1947).

1 Introduction

Consider a Boolean algebra \mathcal{B} of sets. A map $\nu : \mathcal{B} \rightarrow \mathbb{R}^+$ is called a *submeasure* if it satisfies the following properties:

$$\nu(\emptyset) = 0, \tag{1.1}$$

$$A \subset B \implies \nu(A) \leq \nu(B), \tag{1.2}$$

$$A, B \in \mathcal{B} \implies \nu(A \cup B) \leq \nu(A) + \nu(B). \tag{1.3}$$

If we have $\nu(A \cup B) = \nu(A) + \nu(B)$ whenever A and B are disjoint, we say that ν is a (finitely additive) *measure*.

We say that a sequence (E_n) of \mathcal{B} is *disjoint* if $E_n \cap E_m = \emptyset$ whenever $n \neq m$. A submeasure is *exhaustive* if $\lim_{n \rightarrow \infty} \nu(E_n) = 0$ whenever (E_n) is a disjoint sequence of \mathcal{B} . A measure is obviously exhaustive. Given two submeasures ν_1 and ν_2 , we say that ν_1 is *absolutely continuous with respect to* ν_2 if

$$\forall \varepsilon > 0, \exists \alpha > 0, \nu_2(A) \leq \alpha \implies \nu_1(A) \leq \varepsilon. \tag{1.4}$$

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If a submeasure is absolutely continuous with respect to a measure, it is exhaustive. One of the many equivalent forms of Maharam's problem is whether the converse is true.

Maharam's problem: If a submeasure is exhaustive, is it absolutely continuous with respect to a measure?

In words, we are asking whether the only way a submeasure can be exhaustive is because it really resembles a measure. This question has been one of the longest standing classical questions of measure theory. It occurs in a variety of forms (some of which will be discussed below).

Several important contributions were made to Maharam's problem. N. Kalton and J. W. Roberts proved [K-R] that a submeasure is absolutely continuous with respect to a measure if (and, of course, only if) it is uniformly exhaustive, i.e.

$$\forall \varepsilon > 0, \exists n, E_1, \dots, E_n \text{ disjoint} \implies \inf_{i \leq n} \nu(E_i) \leq \varepsilon. \quad (1.5)$$

Thus Maharam's problem can be reformulated as to whether an exhaustive submeasure is necessarily uniformly exhaustive. Two other fundamental contributions by J.W. Roberts [R] and I. Farah [F] are used in an essential way in this paper and will be discussed in great detail later.

We prove that Maharam's problem has negative answer.

Theorem 1.1 *There exists a non-zero exhaustive submeasure ν on the algebra \mathcal{B} of clopen sets of the Cantor set that is not uniformly exhaustive (and thus is not absolutely continuous with respect to a measure). Moreover, no non-zero measure μ on \mathcal{B} is absolutely continuous with respect to ν .*

We now spell out some consequences of Theorem 1.1. It has been known for a while how to deduce these results from Theorem 1.1. For the convenience of the reader these (easy) arguments will be given in a self-contained way in the last section of the paper.

Since Maharam's original question and von Neumann problem are formulated in terms of general Boolean algebras (i.e., that are not a priori represented as algebras of sets) we must briefly mention these. We will denote by 0 and 1 respectively the smallest and the largest element of a Boolean algebra \mathcal{B} , but we will denote the operations by \cap, \cup , etc. as in the case of algebras of sets. A Boolean algebra \mathcal{B} is called σ -complete if any countable set \mathcal{C} has a least upper bound $\cup \mathcal{C}$ (and thus a greatest lower bound $\cap \mathcal{C}$). A submeasure ν on \mathcal{B} is called *continuous* if whenever (A_n) is a decreasing sequence with $\bigcap_n A_n = 0$ we have $\lim_{n \rightarrow \infty} \nu(A_n) = 0$. The submeasure is called *positive* if $\nu(A) = 0 \implies A = 0$.

A σ -complete algebra \mathcal{B} on which there is a positive continuous submeasure is called a *submeasure algebra*. If there is a positive continuous measure on \mathcal{B} , \mathcal{B} is called a *measure algebra*.

Probably the most important consequence of our construction is that it proves the existence of radically new Boolean algebras

Theorem 1.2 *There exists a submeasure algebra \mathcal{B} that is not a measure algebra. In fact, there does not exist a positive measure on \mathcal{B} , and there not exist a (non-zero) continuous measure on \mathcal{B} either.*

This answers a question raised by D. Maharam in her 1947 paper [M].

A subset \mathcal{C} of a boolean algebra \mathcal{B} is called *disjoint* if $A \cap B = \emptyset$ whenever $A, B \in \mathcal{C}$, $A \neq B$. A disjoint set \mathcal{C} is called a *partition* if $\cup \mathcal{C} = 1$ (= the largest element of \mathcal{B}). If every disjoint collection of \mathcal{B} is countable, \mathcal{B} is said to satisfy the *countable chain condition*.

If Π is a partition of \mathcal{B} we say that $A \in \mathcal{B}$ is *finitely covered* by Π if there is a finite subset $\{A_1, \dots, A_n\}$ of Π with $A \subset \bigcup_{i \leq n} A_i$. We say that \mathcal{B} satisfies the *general distributive law* if whenever (Π_n) is a sequence of partitions of \mathcal{B} , there is a single partition Π of \mathcal{B} such that every element of Π is finitely covered by each Π_n . (This terminology is not used by every author, such a σ -algebra is called weakly $(\sigma - \infty)$ distributive in [F1].)

Theorem 1.3 *There exists a σ -complete algebra that satisfies the countable chain condition and the general distributive law, but is not a measure algebra.*

We spell out this statement because it answers negatively a problem raised by J. von Neumann in the Scottish book ([Ma] problem 163), but it is a simple consequence of Theorem 1.2, since every submeasure algebra satisfies the countable chain condition and the general distributive law. Examples of this type had been known under special axioms, such as the negation of Suslin's hypothesis [M], but our example is the first one that does not use any special axiom. (In fact, it has been recently shown [B-J-P], [V], that essentially the only way to produce a counterexample to von Neumann problem that does not use special axioms is indeed to solve Maharam's problem.)

Consider now a topological vector space X with a metrizable topology, and d a translation invariant distance that defines this topology. If \mathcal{B} is a boolean algebra of subsets of a set T , an $(X$ -valued) *vector measure* is a map $\theta : \mathcal{B} \rightarrow X$ such that $\theta(A \cup B) = \theta(A) + \theta(B)$ whenever $A \cap B = \emptyset$. We say that it is *exhaustive* if $\lim_{n \rightarrow \infty} \theta(E_n) = 0$ for each disjoint sequence (E_n) of \mathcal{B} . A positive measure μ on \mathcal{B} is called a *control measure* for θ if

$$\forall \varepsilon > 0, \exists \alpha > 0, \mu(A) \leq \alpha \implies d(0, \theta(A)) \leq \varepsilon.$$

Theorem 1.4 (Negative solution to the Control Measure Problem) *There exists an exhaustive vector-valued measure that does not have a control measure.*

We now explain the organization of the paper. The submeasure we will construct is an object of a rather new nature, since it is very far from being a measure. It is unlikely that a very simple example exists at all, and it should not come as a surprise that our construction is somewhat involved. Therefore it seems necessary to explain first the main ingredients on which the construction relies. The fundamental idea is due to J. W. Roberts [R] and is detailed in Section 2. Another crucial part of the construction is a technical device invented by I. Farah [F]. In Section 3, we produce a kind of “miniature version” of Theorem 1.1, to explain Farah’s device, as well as some of the other main ideas. The construction of ν itself is given in Section 4, and the technical work of proving that ν is not zero and is exhaustive is done in Sections 5 and 6 respectively. Finally, in Section 7 we give the simple (and known) arguments needed to deduce Theorems 1.2 to 1.4 from Theorem 1.1.

Acknowledgments. My warmest thanks go to I. Farah who explained to me the importance of Roberts’s work [R], provided a copy of this hard-to-find paper, rekindled my interest in this problem, and, above all, made an essential technical contribution without which my own efforts could hardly have succeeded.

2 Roberts

Throughout the paper we write

$$T = \prod_{n \geq 1} \{1, \dots, 2^n\}.$$

For $\vec{z} \in T$, we thus have $\vec{z} = (z_n)$, $z_n \in \{1, \dots, 2^n\}$. We denote by \mathcal{B}_n the algebra generated by the coordinates of rank $\leq n$, and $\mathcal{B} = \bigcup_{n \geq 1} \mathcal{B}_n$ the algebra of the clopen sets of T . It is isomorphic to the algebra of the clopen sets of the Cantor set $\{0, 1\}^{\mathbb{N}}$.

We denote by \mathcal{A}_n the set of atoms of \mathcal{B}_n . These are sets of the form

$$\{\vec{z} \in T; z_1 = \tau_1, \dots, z_n = \tau_n\} \quad (2.1)$$

where τ_i is an integer $\leq 2^i$. An element A of \mathcal{A}_n will be called an atom of rank n .

Definition 2.1 [R] *Consider $1 \leq m < n$. We say that a subset X of T is (m, n) -thin if*

$$\forall A \in \mathcal{A}_m, \quad \exists A' \in \mathcal{A}_n, \quad A' \subset A, \quad A' \cap X = \emptyset.$$

In words, in each atom of rank m , X has a hole big enough to contain an atom of rank n . It is obvious that if X is (m, n) -thin, it is also (m, n') -thin when $n' \geq n$.

Definition 2.2 [R] Consider a (finite) subset I of $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We say that $X \subset T$ is I -thin if X is (m, n) -thin whenever $m < n$, $m, n \in I$.

We denote by $\text{card} I$ the cardinality of a finite set I . For two finite sets $I, J \subset \mathbb{N}^*$, we write $I \prec J$ if $\max I \leq \min J$.

The following is implicit in [R] and explicit in [F].

Lemma 2.3 (Roberts's selection lemma). Consider two integers s and t , and sets $I_1, \dots, I_s \subset \mathbb{N}^*$ with $\text{card} I_\ell \geq st$ for $1 \leq \ell \leq s$. Then we can relabel the sets I_1, \dots, I_s so that we can find sets $J_\ell \subset I_\ell$ with $\text{card} J_\ell = t$ and $J_1 \prec J_2 \prec \dots \prec J_s$.

Proof. Let us enumerate $I_\ell = \{i_{1,\ell}, \dots, i_{st,\ell}\}$ where $i_{a,\ell} < i_{b,\ell}$ if $a < b$. We can relabel the sets I_ℓ in order to ensure that

$$\begin{aligned} \forall k \geq 1 \quad i_{t,1} &\leq i_{t,k} \\ \forall k \geq 2 \quad i_{2t,2} &\leq i_{2t,k} \end{aligned}$$

and more generally, for any $\ell < s$ that

$$\forall k \geq \ell \quad i_{\ell t, \ell} \leq i_{\ell t, k} \quad (2.2)$$

We then define

$$J_\ell = \{i_{(\ell-1)t+1, \ell}, \dots, i_{\ell t, \ell}\}.$$

To see that for $1 \leq \ell < s$ we have $J_\ell \prec J_{\ell+1}$ we use (2.2) for $k = \ell + 1$, so that $i_{\ell t, \ell} \leq i_{\ell t, \ell+1} < i_{\ell t+1, \ell+1}$. \square

The reader might observe that it would in fact suffice to assume that $\text{card} I_\ell \geq s(t-1) + 1$; but this refinement yields no benefits for our purposes.

Throughout the paper, given an integer $\tau \leq 2^n$, we write

$$S_{n,\tau} = \{\vec{z} \in T; z_n \neq \tau\} \quad (2.3)$$

so that its complement $S_{n,\tau}^c$ is the set $\{\vec{z} \in T; z_n = \tau\}$. Thus on the set $S_{n,\tau}$ we forbid that the n^{th} coordinate of \vec{z} be τ while on $S_{n,\tau}^c$ we force it to be τ .

Proposition 2.4 Consider sets $X_1, \dots, X_q \subset T$, and assume that for each $\ell \leq q$ the set X_ℓ is I_ℓ -thin, for a certain set I_ℓ with $\text{card} I_\ell \geq 3q$. Then for each n and each integer $\tau \leq 2^n$ we have

$$S_{n,\tau}^c \not\subset \bigcup_{\ell \leq q} X_\ell. \quad (2.4)$$

Proof. We use Lemma 2.3 for $s = q$ and $t = 3$ to produce sets $J_\ell \subset I_\ell$ with $J_1 \prec J_2 \prec \dots \prec J_q$ and $\text{card} J_\ell = 3$. Let $J_\ell = (m_\ell, n_\ell, r_\ell)$, so $r_\ell \leq m_{\ell+1}$ since $J_\ell \prec J_{\ell+1}$.

To explain the idea (on which the paper ultimately relies) let us prove first that $T \not\subset \bigcup_{\ell \leq q} X_\ell$. We make an inductive construction to avoid in turn the sets X_ℓ . We start with any $A_1 \in \mathcal{A}_{m_1}$. Since X_1 is (m_1, n_1) -thin, we can find $C_1 \in \mathcal{A}_{n_1}$ with $C_1 \subset A_1$ and $C_1 \cap X_1 = \emptyset$. Since $n_1 \leq m_2$ we can find $A_2 \in \mathcal{A}_{m_2}$ and $A_2 \subset C_1$, and we continue in this manner. The set C_q does not meet any of the sets X_ℓ .

To prove (2.4), we must ensure that $C_q \cap S_{n,\tau}^c \neq \emptyset$. The fundamental fact is that at each stage we have *two* chances to avoid X_ℓ , using either that X_ℓ is (m_ℓ, n_ℓ) -thin or that it is (n_ℓ, r_ℓ) -thin. The details of the construction depend on the “position” of n with respect to the sets J_ℓ . Rather than enumerating the cases, we explain what happens when $m_1 < n \leq r_1$, and this should make what to do in the other cases obvious.

Case 1. We have $m_1 < n \leq n_1$. Since $S_{n,\tau} \in \mathcal{B}_n \subset \mathcal{B}_{n_1}$, we can choose $A_1 \in \mathcal{A}_{n_1}$ with $A_1 \subset S_{n,\tau}^c$. Since X_1 is (n_1, r_1) -thin, we choose $C_1 \in \mathcal{A}_{r_1}$ with $C_1 \subset A_1$ and $C_1 \cap X_1 = \emptyset$. We then continue as before, choosing $A_2 \subset C_1$, $A_2 \in \mathcal{A}_{m_2}$, etc.

Case 2. We have $m_1 < n_1 < n \leq r_1$. We choose any $A_1 \in \mathcal{A}_{m_1}$. Since X is (m_1, n_1) -thin, we can choose $C_1 \in \mathcal{A}_{n_1}$ with $C_1 \subset A_1$ and $C_1 \cap X_1 = \emptyset$. It is obvious from (2.1) that, since $n_1 < n$, we have $C_1 \cap S_{n,\tau}^c \neq \emptyset$. Since $C_1 \cap S_{n,\tau}^c \in \mathcal{B}_n \subset \mathcal{B}_{r_1} \subset \mathcal{B}_{m_2}$, we can find $A_2 \subset C_1 \cap S_{n,\tau}^c$, $A_2 \in \mathcal{A}_{m_2}$, and we continue as before. \square

Definition 2.5 *Given $\varepsilon > 0$, a submeasure ν on an algebra \mathcal{B} is called ε -exhaustive if for each disjoint sequence (E_n) of \mathcal{B} we have $\limsup_{n \rightarrow \infty} \nu(E_n) \leq \varepsilon$.*

Theorem 2.6 (Roberts) *For each q there exists a submeasure ν on T such that*

$$\forall n, \forall \tau \leq 2^n, \quad \nu(S_{n,\tau}^c) = 1 \quad (2.5)$$

$$\nu \text{ is } \frac{1}{q+1}\text{-exhaustive.} \quad (2.6)$$

Of course, (2.5) implies that ν is not uniformly exhaustive. Let us consider the class \mathcal{C} of subsets X of T that are I -thin (for a set I depending on X) with $\text{card} I \geq 3q$. For $B \in \mathcal{B}$ we define

$$\nu(B) = \min \left(1, \inf \left\{ \frac{1}{q+1} \text{card} F; F \subset \mathcal{C}; B \subset \bigcup F \right\} \right), \quad (2.7)$$

where F runs over the finite subsets of \mathcal{C} and $\bigcup F$ denotes the union of F . It is obvious that ν is a submeasure, and (2.5) is an immediate consequence of Proposition 2.4.

To prove (2.6) it suffices, given a disjoint sequence (E_n) of \mathcal{B} , to prove that $\liminf_{n \rightarrow \infty} \nu(E_n) \leq 1/(1+q)$.

For $X \subset T$, let us write

$$(X)_m = \bigcap \{B \in \mathcal{B}_m; B \supset X\} = \bigcup \{A, A \in \mathcal{A}_m, A \cap X \neq \emptyset\}. \quad (2.8)$$

Since each algebra \mathcal{B}_m is finite, by taking a subsequence we can assume that for some integers $m(n)$ we have $E_n \in \mathcal{B}_{m(n)}$, while

$$\forall k > n, \quad (E_k)_{m(n)} = (E_{n+1})_{m(n)}. \quad (2.9)$$

We claim that for each $k > n + 1$, E_k is $(m(n), m(n+1))$ -thin. To prove this, consider $A \in \mathcal{A}_{m(n)}$. If $A \cap E_k = \emptyset$, any $A' \in \mathcal{A}_{m(n+1)}$ with $A' \subset A$ satisfies $A' \cap E_k = \emptyset$. Otherwise $A \subset (E_k)_{m(n)} = (E_{n+1})_{m(n)}$ by (2.9). Therefore, $E_{n+1} \cap A \neq \emptyset$. Since $E_{n+1} \in \mathcal{B}_{m(n+1)}$, we can find $A' \in \mathcal{A}_{m(n+1)}$ with $A' \subset A$ and $A' \subset E_{n+1}$. But then $A' \cap E_k = \emptyset$ since E_{n+1} and E_k are disjoint. This proves the claim.

It follows that for $n \geq 3q + 1$, E_n is I -thin for $I = (n(1), \dots, n(3q))$ and thus $E_n \in \mathcal{C}$, so that $\nu(E_n) \leq 1/(q+1)$. \square

3 Farah

In [F] I. Farah constructs for each ε an ε -exhaustive submeasure ν that is also pathological, in the sense that every measure that is absolutely continuous with respect to ν is zero. In this paper, we learned several crucial technical ideas, that are essential for our approach. The concepts and the techniques required to prove Proposition 3.5 below are essentially all Farah's.

A class \mathcal{C} of *weighted sets* is a subset of $\mathcal{B} \times \mathbb{R}^+$. For a finite subset $F = \{(X_1, w_1), \dots, (X_n, w_n)\}$ of \mathcal{C} , we write throughout the paper

$$w(F) = \sum_{i \leq n} w_i; \quad \cup F = \bigcup_{i \leq n} X_i, \quad (3.1)$$

and for $B \in \mathcal{B}$ we set

$$\varphi_{\mathcal{C}}(B) = \inf \{w(F); B \subset \cup F\}. \quad (3.2)$$

It is immediate to check that $\varphi_{\mathcal{C}}$ is a submeasure provided $\varphi_{\mathcal{C}}(T) < \infty$. This construction generalizes (2.7). It is generic; for a submeasure ν , we have $\nu = \varphi_{\mathcal{C}}$ where $\mathcal{C} = \{(B, \nu(B)); B \in \mathcal{B}\}$. Indeed, it is obvious that $\varphi_{\mathcal{C}} \leq \nu$, and the reverse inequality follows by subadditivity of ν .

For technical reasons, when dealing with classes of weighted sets, we find it convenient to keep track for each pair (X, w) of a distinguished finite subset I of \mathbb{N}^* . For this reason we define a class of *marked weighted sets* as a subset of $\mathcal{B} \times \mathcal{F} \times \mathbb{R}^+$, where \mathcal{F} denotes the collection of finite subsets of \mathbb{N}^* .

For typographical convenience we write

$$\alpha(k) = \frac{1}{(k+5)^3} \quad (3.3)$$

and we consider a sequence $(N(k))$ that be specified later. The specific choice is anyway completely irrelevant, what matters is that this sequence increases fast enough. In fact, there is nothing magic about the choice of $\alpha(k)$ either. Any sequence such that $\sum_k k\alpha(k) < \infty$ would do. We like to stress that none of the numerical quantities occurring in our construction plays an essential role. These are all simple choices that are made for convenience. No attempts whatsoever have been made to make optimal or near optimal choices. Let us also point out that for the purpose of the present section it would work just fine to take $\alpha(k) = (k+5)^{-1}$, and that the reasons for taking a smaller value will become clear only in the next section. For $k \geq 1$ we define the class \mathcal{D}_k of marked weighted sets by

$$\mathcal{D}_k = \left\{ (X, I, w); \exists (\tau(n))_{n \in I}, X = \bigcap_{n \in I} S_{n, \tau(n)}; \text{card} I \leq N(k), \right. \\ \left. w = 2^{-k} \left(\frac{N(k)}{\text{card} I} \right)^{\alpha(k)} \right\}. \quad (3.4)$$

The most important part of \mathcal{D}_k consists of the triplets (X, I, w) where $\text{card} I = N(k)$ and $w = 2^{-k}$. The purpose of the relation $w = 2^{-k} (N(k)/\text{card} I)^{\alpha(k)}$ is to allow the crucial Lemma 3.1 below. To understand the relation between the different classes \mathcal{D}_k it might help to observe the following. Whenever X and I are as in (3.4) and whenever $N(k) \geq \text{card} I$ we have $(X, I, w_k) \in \mathcal{D}_k$ for $w_k = 2^{-k} (N(k)/\text{card} I)^{\alpha(k)}$. If we assume, as we may, that the sequence $2^{-k} N(k)^{\alpha(k)}$ increases, we see that the sequence (w_k) increases. It is then the smallest value possible of k that gives the smallest possible value of w_k , which, as will be apparent in the formula (3.7) below is the only one that matters.

Given a subset J of \mathbb{N}^* we say that a subset X of T *depends only on the coordinates of rank* $n \in J$ if whenever $\vec{z}, \vec{z}' \in T$ are such that $z_n = z'_n$ for $n \in J$, we have $\vec{z} \in T$ iff $\vec{z}' \in T$. Equivalently, we sometimes say that such a set does not depend on the coordinates of rank $n \in J^c = \mathbb{N}^* \setminus J$. One of the key ideas of the definition of \mathcal{D}_k is the following simple fact.

Lemma 3.1 *Consider $(X, I, w) \in \mathcal{D}_k$ and $J \subset \mathbb{N}^*$. Then there is $(X', I', w') \in \mathcal{D}_k$ such that $X \subset X'$, X' depends only on the coordinates in J and*

$$w' = w \left(\frac{\text{card} I}{\text{card} I \cap J} \right)^{\alpha(k)}. \quad (3.5)$$

Since $\alpha(k)$ is small, w' is not really larger than w unless $\text{card} I \cap J \ll \text{card} I$. In particular, since $\alpha(k) \leq 1/2$ we have

$$\text{card} I \cap J \geq \frac{1}{4} \text{card} I \implies w' \leq 2w. \quad (3.6)$$

Proof. We define (X', I', w') by (3.5) and $I' = I \cap J$,

$$X' = \bigcap_{n \in I'} S_{n, \tau(n)},$$

where $\tau(n)$ is as in (3.4). □

A class of marked weighted sets is a subset of $\mathcal{B} \times \mathcal{F} \times \mathbb{R}^+$. By projection onto $\mathcal{B} \times \mathbb{R}^+$, to each class \mathcal{C} of marked weighted sets, we can associate a class \mathcal{C}^* of weighted sets. For a class \mathcal{C} of marked weighted sets, we then define $\varphi_{\mathcal{C}}$ as $\varphi_{\mathcal{C}^*}$ using (3.2). As there is no risk of confusion, we will not distinguish between \mathcal{C} and \mathcal{C}^* at the level of notation. We define

$$\mathcal{D} = \bigcup_{k \geq 1} \mathcal{D}_k; \quad \psi = \varphi_{\mathcal{D}}. \quad (3.7)$$

Proposition 3.2 *Let us assume that*

$$N(k) \geq 2^{k+6} (2^{k+5})^{1/\alpha(k)}. \quad (3.8)$$

Then $\psi(T) \geq 2^5$. Moreover ψ is pathological in the sense that if a measure μ on \mathcal{B} is absolutely continuous with respect to ψ , then $\mu = 0$.

Pathological submeasures seem to have been constructed first implicitly in [D-R] and explicitly in [P].

Proof. To prove that $\psi(T) \geq 2^5$, we consider a finite subset F of \mathcal{D} , with $w(F) < 2^5$, and we prove that $T \not\subset \cup F$. For $k \geq 1$ let $F_k = F \cap \mathcal{D}_k$. For $(X, I, w) \in \mathcal{D}_k$, we have $w \geq 2^{-k}$, so that $\text{card} F_k \leq 2^{k+5}$ since $w(F_k) \leq w(F) < 2^5$. Also we have

$$2^{-k} \left(\frac{N(k)}{\text{card} I} \right)^{\alpha(k)} = w \leq w(F) \leq 2^5,$$

so that $\text{card} I \geq (2^{k+5})^{-1/\alpha(k)} N(k) := c(k)$. Thus, under (3.8) we have $c(k) \geq 2^{k+6}$. Let us enumerate F as a sequence $(X_r, I_r, w_r)_{r \leq r_0}$ (where $r_0 = \text{card} F$) in such a way that if $(X_r, I_r, w_r) \in F_{k(r)}$, the sequence $k(r)$ is non-decreasing. Since

$$\sum_{\ell < k} \text{card} F_\ell \leq \sum_{\ell < k} 2^{\ell+5} < 2^{k+5},$$

we see that $r \geq 2^{k+5}$ implies $k(r) \geq k$ and thus $\text{card} I_r \geq c(k)$. If we assume (3.8) we then see that $\text{card} I_r \geq r+1$. Indeed this is true if $r < 2^6$ because $\text{card} I \geq c(1) \geq 2^6$, and if $r \geq 2^6$ and if k is the largest integer with $r \geq 2^{k+5}$, then $c(k) \geq 2^{k+6} \geq r+1$. We can then pick inductively integers $i_r \in I_r$ that are all different. If $X_r = \bigcap_{n \in I_r} S_{n, \tau_r(n)}$, any \vec{z} in T with $z_{i_r} = \tau_r(i_r)$ for $r \leq r_0$ does not belong to any of the sets X_r , and thus $\cup F \neq T$. This proves that $\psi(T) \geq 2^5$.

We prove now that ψ is pathological. Consider a measure μ on \mathcal{B} , and assume that there exists k such that

$$\psi(B) \leq 2^{-k} \implies \mu(B) \leq \varepsilon.$$

For each $\vec{\tau} = (\tau(n))_{n \leq N(k)}$, we consider the set

$$X_{\vec{\tau}} = \bigcap_{n \leq N(k)} S_{n, \tau(n)}$$

so that if $I = \{1, \dots, N(k)\}$ we have $(X_{\vec{\tau}}, I, 2^{-k}) \in \mathcal{D}_k$ and thus $\psi(X_{\vec{\tau}}) \leq 2^{-k}$, and hence $\mu(X_{\vec{\tau}}) \leq \varepsilon$.

Let us denote by Av the average over all values of $\vec{\tau}$, so that

$$\int \text{Av}(1_{X_{\vec{\tau}}}(\vec{z})) d\mu(\vec{z}) = \text{Av} \int 1_{X_{\vec{\tau}}}(\vec{z}) d\mu(\vec{z}) = \text{Av} \mu(X_{\vec{\tau}}) \leq \varepsilon. \quad (3.9)$$

It should be clear that the quantity $\text{Av}(1_{X_{\vec{\tau}}}(\vec{z}))$ is independent of \vec{z} . Its value a_k satisfies

$$a_k = \int \text{Av} 1_{X_{\vec{\tau}}}(\vec{z}) d\lambda(\vec{z}) = \text{Av} \int 1_{X_{\vec{\tau}}}(\vec{z}) d\lambda(\vec{z})$$

where λ denotes the uniform measure on T . Now

$$\int 1_{X_{\vec{\tau}}}(\vec{z}) d\lambda(\vec{z}) = \lambda(X_{\vec{\tau}}) = \prod_{n \leq N(k)} (1 - 2^{-n})$$

is bounded below independently of k , so that a_k is bounded below independently of k . Finally (3.9) yields

$$\varepsilon \geq \int \text{Av}(1_{X_{\vec{\tau}}}(\vec{z})) d\mu(\vec{z}) = a_k \mu(T),$$

and since ε is arbitrary this shows that $\mu(T) = 0$. \square

Theorem 3.3 *If the sequence $N(k)$ is chosen as in (3.8), for each $\varepsilon > 0$ we can find an ε -exhaustive submeasure $\nu \leq \psi$.*

This result is of course much weaker than Theorem 1.1. We present its proof for pedagogical reasons. Several of the key ideas required to prove Theorem 1.1 will already be needed here, and should be much easier to grasp in this simpler setting.

Given $A \in \mathcal{A}_m$, let us define the map $\pi_A : T \rightarrow A$ as follows: If τ_1, \dots, τ_m are such that

$$\vec{z} \in A \iff \forall i \leq m, \quad z_i = \tau_i$$

then for $\vec{z} \in T$ we have $\pi_A(\vec{z}) = \vec{y}$ where

$$\vec{y} = (\tau_1, \dots, \tau_m, z_{m+1}, \dots).$$

Definition 3.4 (Farah) *Given $m < n$, we say that a set $X \subset T$ is (m, n, ψ) -thin if*

$$\forall A \in \mathcal{A}_m, \quad \exists C \in \mathcal{B}_n, \quad C \subset A, \quad C \cap X = \emptyset, \quad \psi(\pi_A^{-1}(C)) \geq 1.$$

The idea is now that in each atom of rank m , X has a \mathcal{B}_n -measurable hole that is large with respect to ψ . Of course, we cannot require that $\psi(C) \geq 1$ because $\psi(C) \leq \psi(A)$ will be small, and one should think of $\psi(\pi_A^{-1}(C))$ as measuring the “size of C with respect to A ”.

Obviously, if $n' \geq n$ and if X is (m, n, ψ) -thin, it is also (m, n', ψ) -thin. For a subset I of \mathbb{N}^* , we say that X is (I, ψ) -thin if it is (m, n, ψ) -thin whenever $m, n \in I$, $m < n$. By the previous observation, it suffices that this should be the case when m and n are consecutive elements of I .

Consider a given integer q and consider an integer b , to be determined later. Consider the class \mathcal{F} of marked weighted sets defined as

$$\mathcal{F} = \{(X, I, w); X \text{ is } (I, \psi)\text{-thin, } \text{card} I = b, w = 2^{-q}\}.$$

We define

$$\nu = \varphi_{\mathcal{F} \cup \mathcal{D}},$$

where \mathcal{D} is the class (2.9). Thus $\nu \leq \psi = \varphi_{\mathcal{D}}$, so it is pathological.

Proposition 3.5 *The submeasure ν is 2^{-q} -exhaustive.*

Proposition 3.6 *If we assume*

$$b = 2^{2q+10} \tag{3.10}$$

we have $\nu(T) \geq 2^4$.

Both these results assume that (3.8) holds. This condition is assumed without further mention in the rest of the paper.

We first prove Proposition 3.5. Again, the arguments are due to I. Farah [F] and are of essential importance.

Lemma 3.7 *Consider a sequence $(E_i)_{i \geq 1}$ of \mathcal{B} and assume that*

$$\forall n, \quad \psi \left(\bigcup_{i \leq n} E_i \right) < 1.$$

Assume that for a certain $m \geq 1$, the sets E_i do not depend on the coordinates of rank $\leq m$. Then for each $\alpha > 0$ we can find a set $C \in \mathcal{B}$, that does not depend on the coordinates of rank $\leq m$, and satisfies $\psi(C) \leq 2$ and

$$\forall i \geq 1, \quad \psi(E_i \setminus C) \leq \alpha.$$

Proof. By definition of ψ for each n we can find a finite set $F_n \subset \mathcal{D}$ with $w(F_n) < 1$ and $\bigcup_{i \leq n} E_i \subset \cup F_n$. For an integer $r \geq m + 2$, let

$$\begin{aligned} F_n^r = \{ (X, I, w) \in F_n; \text{card} I \cap \{m+1, \dots, r-1\} < \text{card} I / 2; \\ \text{card} I \cap \{m+1, \dots, r\} \geq \text{card} I / 2 \}, \end{aligned} \quad (3.11)$$

so that the sets F_k^r are disjoint as r varies. We use Lemma 3.1 and (3.6) with $J = I \cap \{m+1, \dots, r\}$ to obtain for each (X, I, w) an element (X', I', w') of \mathcal{D} such that $X' \supset X$, $w' \leq 2w$, and X' depends only on the coordinates of rank in $\{m+1, \dots, r\}$ (or, equivalently, $I \subset \{m+1, \dots, r\}$). We denote by $F_n'^r$ the collection of the sets (X', I', w') as $(X, I, w) \in F_n^r$. Thus $\cup F_n'^r \supset \cup F_n^r$, and $w(F_n'^r) \leq 2w(F_n^r)$.

Consider an integer i , and j such that $E_i \in \mathcal{B}_j$. We prove that for $n \geq i$ we have $E_i \subset \bigcup_{r \leq j} \cup F_n'^r$. Otherwise, since both these sets depend only on the coordinates of rank in $\{m+1, \dots, j\}$, we can find a set A depending only on those coordinates with $A \subset E_i \setminus \bigcup_{r \leq j} \cup F_n'^r$, and thus $A \subset E_i \setminus \bigcup_{r \leq j} \cup F_n^r$. Since $E_i \subset \cup F_n$, we have $A \subset \cup F^\sim$, where $F^\sim = F_n \setminus \bigcup_{r \leq j} F_n^r$. Now, by definition of F_n^r , if $(X, I, w) \in F^\sim$, we have $\text{card}(I \setminus \{m+1, \dots, j\}) \geq \text{card} I / 2$. Again use Lemma 3.1, now with $J = \{m+1, \dots, j\}^c$ to see that we can find (X', I', w') in \mathcal{D} with $w' \leq 2w$ and $X' \supset X$, X' does not depend on the coordinates of rank in $\{m+1, \dots, j\}$. Let F' be the collection of these triplets (X', I', w') , so $F' \subset \mathcal{D}$ and $w(F') \leq 2w(F_n) \leq 2$. Now $\cup F' \supset \cup F^\sim \supset A$, and since $\cup F'$ does not depend on the coordinates in $\{m+1, \dots, r\}$, while A is determined by these coordinates, we have $\cup F' = T$. But this would imply that $\psi(T) \leq 2$, while we have proved that $\psi(T) \geq 2^5$.

Thus $E_i \subset \bigcup_{r \leq j} \cup F_n'^r$. For (X, I, w) in $F_n'^r$, we have $I \subset \{m+1, \dots, r\}$. Under (3.8) we have that if $(X, I, w) \in \mathcal{D}_k \cap F_n'^r$ then

$$w = 2^{-k} \left(\frac{N(k)}{\text{card} I} \right)^{\alpha(k)} \geq \frac{2^5}{\text{card} I^{\alpha(k)}} \geq \frac{2^5}{r^{\alpha(k)}}, \quad (3.12)$$

which shows (since $w(F_n^r) \leq 1$) that k remains bounded independently of n . Since moreover $I \subset \{m+1, \dots, r\}$ there exists a finite set $\mathcal{D}^r \subset \mathcal{D}$ such that $F_n^r \subset \mathcal{D}^r$ for all n . Then, by taking a subsequence if necessary, we can assume that for each r the sets F_n^r are eventually equal to a set F^r . For each triplet (X, I, w) in F^r , the set X depends only on the coordinates of rank in $\{m+1, \dots, r\}$, and it should be obvious that $\sum_{r \geq m} w(F^r) \leq 2$ and $E_i \subset \bigcup_{r \leq j} \cup F^r$ (whenever j is such that $E_i \in \mathcal{B}_j$).

Consider r_0 such that $\sum_{r > r_0} w(F^r) \leq \alpha$, and let $C = \bigcup_{r \leq r_0} \cup F^r$. Thus $C \in \mathcal{B}$, C does not depend on the coordinates of rank $\leq m$ and $\psi(C) \leq \sum_{r \leq r_0} w(F^r) \leq 2$. Moreover, since $E_i \subset \bigcup_{r \leq j} \cup F^r$ whenever j is large enough that $E_i \in \mathcal{B}_j$, we have

$$E_i \setminus C \subset \bigcup_{r_0 < r \leq j} \cup F^r,$$

so that $\psi(E_i \setminus C) \leq \sum_{r > r_0} w(F^r) \leq \alpha$. \square

Lemma 3.8 (Farah) *Consider $\alpha > 0$, $B \in \mathcal{B}_m$, and a disjoint sequence (E_i) of \mathcal{B} . Then there exists $n > m$, a set $B' \subset B$, $B' \in \mathcal{B}_n$, so that B' is (m, n, ψ) -thin and $\limsup_{i \rightarrow \infty} \psi((B \cap E_i) \setminus B') \leq \alpha$.*

Proof. Consider $\alpha' = \alpha / \text{card} \mathcal{A}_m$. Consider $A \in \mathcal{A}_m$, $A \subset B$.

Case 1. $\exists p; \psi \left(\pi_A^{-1} \left(\bigcup_{i \leq p} E_i \right) \right) \geq 1$.

We set $C' = C'(A) = A \setminus \bigcup_{i \leq p} E_i$, so that $\psi(\pi_A^{-1}(A \setminus C')) \geq 1$ and $(A \cap E_i) \setminus C' = \emptyset$ for $i > p$.

Case 2. $\forall p; \psi \left(\pi_A^{-1} \left(\bigcup_{i \leq p} E_i \right) \right) < 1$.

The sets $\pi_A^{-1}(E_i)$ do not depend on the coordinates of rank $\leq m$ so by Lemma 3.7 we can find a set $C \in \mathcal{B}$, that does not depend on the coordinates of rank $\leq m$, with $\psi(C) \leq 2$ and $\limsup_{i \rightarrow \infty} \psi(\pi_A^{-1}(E_i) \setminus C) \leq \alpha'$. Let $C' = C'(A) = \pi_A(C) \subset A$. Since C does not depend on the coordinates of rank $\leq m$, we have $C = \pi_A^{-1}(C')$ so that $\psi(\pi_A^{-1}(C')) \leq 2$. Since $\pi_A(\vec{z}) = \vec{z}$ for $\vec{z} \in A$, we have

$$(A \cap E_i) \setminus C' \subset \pi_A^{-1}(E_i) \setminus C$$

so that

$$\limsup_{i \rightarrow \infty} \psi((A \cap E_i) \setminus C') \leq \limsup_{i \rightarrow \infty} \psi(\pi_A^{-1}(E_i) \setminus C) \leq \alpha'.$$

Let us now define

$$B' = \bigcup \{C' = C'(A); A \in \mathcal{A}_m, A \subset B\},$$

so that

$$\limsup_{i \rightarrow \infty} \psi((B \cap E_i) \setminus B') \leq \sum \limsup_{i \rightarrow \infty} \psi((A \cap E_i) \setminus C') \leq \alpha' \text{card} \mathcal{A}_m \leq \alpha, \quad (3.13)$$

where the summation is over $A \subset B$, $A \in \mathcal{A}_m$.

Consider n such that $B' \in \mathcal{B}_n$. To prove that B' is (m, n, ψ) -thin it suffices to prove that $\psi(\pi_A^{-1}(A \setminus C')) \geq 1$ whenever $A \in \mathcal{A}_m$, $A \subset B$, because $B' \cap A = C'$, and thus $A \setminus B' = A \setminus C'$. This was already done in case 1. In case 2, we observe that

$$\psi(\pi_A^{-1}(A \setminus C')) = \psi(\pi_A^{-1}(C')^c)$$

and that

$$2^5 \leq \psi(T) \leq \psi(\pi_A^{-1}(C')) + \psi(\pi_A^{-1}(C')^c) \leq 2 + \psi(\pi_A^{-1}(C')^c).$$

□

Proof of Proposition 3.5 (Farah). Consider a disjoint sequence $(E_i)_{i \geq 1}$ of \mathcal{B} . Consider $\alpha > 0$. Starting with $B_0 = T$, we use Lemma 3.8 to recursively construct sets $B_\ell \in \mathcal{B}$ and integers (n_1, n_2, \dots) such that B_ℓ is (I_ℓ, ψ) -thin for $I_\ell = \{1, n_1, n_2, \dots, n_\ell\}$ and $B_\ell \subset B_{\ell-1}$,

$$\limsup_{i \rightarrow \infty} \psi((E_i \cap B_{\ell-1}) \setminus B_\ell) \leq \alpha. \quad (3.14)$$

We have, since $B_0 = T$

$$E_i \setminus B_\ell \subset \bigcup_{m \leq \ell} ((E_i \cap B_{m-1}) \setminus B_m),$$

and the subadditivity of ψ then implies that

$$\psi(E_i \setminus B_\ell) \leq \sum_{m \leq \ell} \psi((E_i \cap B_{m-1}) \setminus B_m)$$

and thus

$$\limsup_{i \rightarrow \infty} \psi(E_i \setminus B_\ell) \leq \alpha \ell. \quad (3.15)$$

For $\ell = b$ (or even $\ell = b - 1$) (where b is given by (3.10)) the definition of \mathcal{F} shows that $(B_\ell, I_\ell, 2^{-q}) \in \mathcal{F}$, and thus $\nu(B_\ell) \leq 2^{-q}$. Since $\nu \leq \psi$, we have

$$\nu(E_i) \leq \nu(B_\ell) + \psi(E_i \setminus B_\ell) \leq 2^{-q} + \psi(E_i \setminus B_\ell),$$

and (3.15) shows that

$$\limsup_{i \rightarrow \infty} \nu(E_i) \leq 2^{-q} + \alpha \ell.$$

Since α is arbitrary, the proof is complete. \square

We turn to the proof of Proposition 3.6. Considering $F_1 \subset \mathcal{F}$ and $F_2 \subset \mathcal{D}$, we want to show that

$$w(F_1) + w(F_2) < 2^4 \implies T \not\subset (\cup F_1) \cup (\cup F_2).$$

Since $w \geq 2^{-q}$ for $(X, I, w) \in \mathcal{F}$, we have $w(F_1) \geq 2^{-q} \text{card} F_1$, so that $\text{card} F_1 \leq 2^{q+4}$. We appeal to Lemma 2.3 with $s = \text{card} F_1$ and $t = b2^{-q-4}$ (which is an integer by (3.10)) to see that we can enumerate $F_1 = (X_\ell, I_\ell, w_\ell)_{\ell \leq s}$ and find sets $J_1 \prec J_2 \prec \dots \prec J_s$ with $\text{card} J_\ell = t$ and $J_\ell \subset I_\ell$.

Let us enumerate

$$J_\ell = \{i_{1,\ell}, \dots, i_{t,\ell}\}. \quad (3.16)$$

An essential idea is that each of the pairs $\{i_{u,\ell}, i_{u+1,\ell}\}$ for $1 \leq u \leq t-1$ gives us a chance to avoid X_ℓ . We are going for each ℓ to choose one of these chances using a counting argument. For

$$\vec{u} = (u(\ell))_{\ell \leq s} \in \{1, \dots, t-1\}^s, \quad (3.17)$$

we define the set

$$W(\vec{u}) = \bigcup_{\ell \leq s}]i_{u(\ell),\ell}, i_{u(\ell)+1,\ell}],$$

where for integers $m < n$ we define $]m, n] = \{m+1, \dots, n\}$.

We consider the quantity

$$S(\vec{u}) = \sum \{w; (X, I, w) \in F_2, \text{card}(I \cap W(\vec{u})) \geq \text{card} I / 2\}.$$

We will choose \vec{u} so that $S(\vec{u})$ is small. Let us denote by Av the average over all possible choices of \vec{u} . Then, for any set I , by linearity of Av , we have

$$\begin{aligned} \text{Av}(\text{card}(I \cap W(\vec{u}))) &= \sum_{\ell \leq s} \text{Av}(\text{card}(I \cap]i_{u(\ell),\ell}, i_{u(\ell)+1,\ell}])) \\ &= \sum_{\ell \leq s} \frac{1}{t-1} \text{card}(I \cap]i_{1,\ell}, i_{t,\ell}]) \leq \frac{1}{t-1} \text{card} I. \end{aligned}$$

Thus, by Markov's inequality, we have

$$\text{Av}(1_{\{\text{card}(I \cap W(\vec{u})) \geq \text{card} I / 2\}}) \leq \frac{2}{t-1}$$

and, using linearity of average, we get

$$\text{Av}(S(\vec{u})) \leq \frac{2}{t-1} w(F_2) \leq \frac{2^5}{t-1} \leq \frac{2^{q+10}}{b}.$$

Thus, we can find \vec{u} such that $S(\vec{u}) \leq 2^{q+10}/b$. We fix this value of \vec{u} once and for all. To lighten notation we set

$$W = W(\vec{u}); \quad m_\ell = i_{u(\ell), \ell}, \quad n_\ell = i_{u(\ell)+1, \ell}, \quad W_\ell =]m_\ell, n_\ell] \quad (3.18)$$

so that $W = \bigcup_{\ell \leq s} W_\ell$, and $n_\ell \leq m_{\ell+1}$ since $n_\ell \in J_\ell$, $m_{\ell+1} \in J_{\ell+1}$, $J_\ell \prec J_{\ell+1}$.

Let us define

$$F_3 = \{(X, I, w) \in F_2; \text{card}(I \cap W) \geq \text{card} I / 2\} \quad (3.19)$$

$$F_4 = \{(X, I, w) \in F_2; \text{card}(I \cap W) < \text{card} I / 2\}, \quad (3.20)$$

so that $F_2 = F_3 \cup F_4$, and the condition $S(\vec{u}) \leq 2^{q+10}/b$ means that

$$w(F_3) \leq \frac{2^{q+10}}{b}.$$

In particular if $(X, I, w) \in F_3$ we have $w \leq 2^{q+10}/b$. Since $w \geq 2^{-k}$ for $(X, I, w) \in \mathcal{D}_k$ we see that under (3.10) we have

$$(X, I, w) \in \mathcal{D}_k \cap F_3 \implies k \geq q. \quad (3.21)$$

Since $s = \text{card} F_1 \leq 2^{q+4}$ and $W = \bigcup_{\ell \leq s} W_\ell$, if $\text{card}(I \cap W) \geq \text{card} I / 2$, there must exist $\ell \leq s$ with $\text{card}(I \cap W_\ell) \geq 2^{-q-5} \text{card} I$. This shows that if we define

$$F_3^\ell = \{(X, I, w) \in F_3; \text{card}(I \cap W_\ell) \geq 2^{-q-5} \text{card} I\}, \quad (3.22)$$

then we have $F_3 = \bigcup_{\ell \leq s} F_3^\ell$.

We appeal to Lemma 3.1 with $J = W_\ell$, using the fact that if $k \geq q$ we have

$$(2^{q+5})^{\alpha(k)} \leq 2$$

(with huge room to spare!), to find for each $(X, I, w) \in F_3^\ell$ a triplet $(X', I', w') \in \mathcal{D}$ with $X \subset X'$, $w' \leq 2w$, such that X' depends only on the coordinates of rank in W_ℓ . Let $F_3'^\ell$ be the collection of these triplets, so that under (3.10) we have

$$w(F_3'^\ell) \leq 2w(F_3^\ell) \leq 2w(F_3) \leq \frac{2^5}{b} \leq \frac{1}{2}.$$

We use again Lemma 3.1, this time for J the complement of W , so that $\text{card}(I \cap J) \geq \text{card} I / 2$ for $(X, I, w) \in F_4$, and we can find $(X', I', w) \in \mathcal{D}$ with $w' \leq 2w$, X' contains

X and depends only on coordinates whose rank is not in W . Let F'_4 be the collection of these triplets, so that $w(F'_4) \leq 2w(F_4) < 2^5$.

Since $\psi(T) \geq 2^5$, we have $T \not\subset \cup F'_4$, so that we can find $\vec{z} \in T \setminus \cup F'_4$. Since $\cup F'_4$ depends only on the coordinates whose rank is not in W , if $\vec{z}' \in T$ is such that $z_i = z'_i$ for $i \notin W$, then $\vec{z}' \notin \cup F'_4$. To conclude the proof, we are going to construct such a \vec{z}' that does not belong to any of the sets X_ℓ or $\cup F'_3{}^\ell$. (Thus \vec{z}' will not belong to $(\cup F_1) \cup (\cup F_2)$.) First, let $A_1 \in \mathcal{A}_{m_1}$ such that $\vec{z} \in A_1$. Since X_1 is (m_1, n_1, ψ) -thin, there exists $C \in \mathcal{B}_{n_1}$, $C \cap X_1 = \emptyset$, $\psi(\pi_{A_1}^{-1}(C)) \geq 1$. Since $w(F_3{}^1) \leq 1/2$, we therefore have $\pi_{A_1}^{-1}(C) \setminus C' \neq \emptyset$, where $C' = \cup F_3{}^1$. Since C' does not depend on the coordinates of rank $\leq m_1$ we have $C' = \pi_{A_1}^{-1}(C')$, so that $\pi_{A_1}^{-1}(C) \setminus \pi_{A_1}^{-1}(C') \neq \emptyset$, and hence $C \setminus C' \neq \emptyset$. Since C' depends only on the coordinates of rank in W_1 , we have $C' \in \mathcal{B}_{n_1}$, and since $C \in \mathcal{B}_{n_1}$, we can find $A' \in \mathcal{A}_{n_1}$ with $A' \subset C \setminus C'$, so that $A' \cap X_1 = \emptyset$ and $A' \cap \cup F_3{}^1 = \emptyset$. Next, we find $A_2 \in \mathcal{A}_{m_2}$ with $A_2 \subset A'$ such that if $\vec{y} \in A_2$ then

$$\forall i, \quad n_1 < i \leq m_2 \implies y_i = z_i,$$

and we continue the construction in this manner. \square

4 The construction

Given an integer p , we will make a construction “with p levels”, and we will then take a kind of limit as $p \rightarrow \infty$. We consider the sequence $\alpha(k)$ as in (3.3), and a sequence $M(k)$ to be specified later. The only requirement is that this sequence increases fast enough. We recall the class \mathcal{D} constructed in the previous section.

We construct classes $(\mathcal{E}_{k,p})_{k \leq p}$, $(\mathcal{C}_{k,p})_{k \leq p}$ of marked weighted sets, and submeasures $(\varphi_{k,p})_{k \leq p}$ as follows. First, we set

$$\begin{aligned} \mathcal{C}_{p,p} &= \mathcal{E}_{p,p} = \mathcal{D} \\ \varphi_{p,p} &= \varphi_{\mathcal{D}} = \psi. \end{aligned}$$

Having defined $\varphi_{k+1,p}$, $\mathcal{E}_{k+1,p}$, $\mathcal{C}_{k+1,p}$, we then set

$$\begin{aligned} \mathcal{E}_{k,p} &= \left\{ (X, I, w); \quad X \in \mathcal{B}, \quad X \text{ is } (I, \varphi_{k+1,p})\text{-thin,} \quad \text{card} I \leq M(k), \right. \\ &\quad \left. w = 2^{-k} \left(\frac{M(k)}{\text{card} I} \right)^{\alpha(k)} \right\} \\ \mathcal{C}_{k,p} &= \mathcal{C}_{k+1,p} \cup \mathcal{E}_{k,p} \\ \varphi_{k,p} &= \varphi_{\mathcal{C}_{k,p}}. \end{aligned}$$

To take limits, we fix an ultrafilter \mathcal{U} on \mathbb{N}^* and we define the class \mathcal{E}_k of marked weighted sets by

$$(X, I, w) \in \mathcal{E}_k \iff \{p; (X, I, w) \in \mathcal{E}_{k,p}\} \in \mathcal{U} \quad (4.1)$$

Of course, one can also work with subsequences if one so wishes. It seems plausible that with further effort one might prove that $(X, I, w) \in \mathcal{E}_k$ if and only if $(X, I, w) \in \mathcal{E}_{k,p}$ for all p large enough, but this fact, if true, is not really relevant for our main purpose.

We define

$$\mathcal{C}_k = \mathcal{D} \cup \bigcup_{\ell \geq k} \mathcal{E}_\ell = \mathcal{C}_{k+1} \cup \mathcal{E}_k; \quad \nu_k = \varphi_{\mathcal{C}_k}; \quad \nu = \nu_1.$$

Let us assume that

$$M(k) \geq 2^{(k+5)/\alpha(k)}. \quad (4.2)$$

Then if $w < 2^5$ and $(X, I, w) \in \mathcal{E}_{r,p}$, since

$$w = 2^{-r} \left(\frac{M(r)}{\text{card} I} \right)^{\alpha(r)} \geq \frac{2^5}{\text{card} I^{\alpha(r)}}, \quad (4.3)$$

r remains bounded independently of p . It then follows from (4.1) that if $w < 2^5$ we have

$$(X, I, w) \in \mathcal{C}_k \iff \{p; (X, I, w) \in \mathcal{C}_{k,p}\} \in \mathcal{U}. \quad (4.4)$$

Theorem 4.1 *We have $\nu(T) > 0$, ν is exhaustive, ν is pathological, and ν is not uniformly exhaustive.*

The hard work will of course be to show that $\nu(T) > 0$ and that ν is exhaustive, but the other two claims are easy. Since $\nu \leq \psi$, it follows from Proposition 3.2 that ν is pathological. It then follows from the Kalton-Roberts theorem that ν is not uniformly exhaustive. This can also be seen directly by showing that $\liminf_{n \rightarrow \infty} \inf_{\tau \leq 2^n} \nu(S_{n,\tau}^c) > 0$. To see this, consider $I \subset \mathbb{N}^*$, and for $n \in I$ let $\tau(n) \leq 2^n$. Then

$$T \subset \bigcup_{n \in I} S_{n,\tau(n)}^c \cup \left(\bigcap_{n \in I} S_{n,\tau(n)} \right)$$

so that by subadditivity we have

$$\begin{aligned} 1 \leq \nu(T) &\leq \sum_{n \in I} \nu(S_{n,\tau(n)}^c) + \nu \left(\bigcap_{n \in I} S_{n,\tau(n)} \right) \\ &\leq \sum_{n \in I} \nu(S_{n,\tau(n)}^c) + \psi \left(\bigcap_{n \in I} S_{n,\tau(n)} \right). \end{aligned}$$

The definition of \mathcal{D} shows that if $\text{card} I = N(1)$, the last term is $\leq 1/2$, and thus $\sum_{n \in I} \nu(S_{n, \tau(n)}^c) \geq 1/2$. This proves that ν is not uniformly exhaustive.

It could be of interest to observe that the submeasure ν has nice invariant properties. For each n it is invariant under any permutation of the elements of T_n . It was observed by Roberts [R] that if there exists an exhaustive submeasure that is not uniformly exhaustive, this submeasure can be found with the above invariance property. This observation was very helpful to the author. It pointed to what should be a somewhat canonical example.

5 The main estimate

Before we can say anything at all about ν , we must of course control the submeasures $\varphi_{k,p}$. Let us define

$$c_1 = 2^4, \quad c_{k+1} = c_k 2^{2\alpha(k)}$$

so that since $\sum_{k \geq 1} \alpha(k) \leq 1/2$ we have

$$c_k \leq 2^5. \tag{5.1}$$

Theorem 5.1 *Assume that the sequence $M(k)$ satisfies*

$$M(k) \geq 2^{2k+10} 2^{(k+5)/\alpha(k)} (2^3 + N(k-1)). \tag{5.2}$$

Then

$$\forall p, \forall k \leq p, \quad \varphi_{k,p}(T) \geq c_k. \tag{5.3}$$

Of course (5.2) implies (4.2). It is the only requirement we need on the sequence $(M(k))$.

The proof of Theorem 5.1 resembles that of Proposition 3.6. The key fact is that the class $\mathcal{E}_{k,p}$ has to a certain extent the property of \mathcal{D}_k stressed in Lemma 3.1, at least when the set J is not too complicated.

The following lemma expresses such a property when J is an interval. We recall the notation $(X)_n$ of (2.8).

Lemma 5.2 *Consider $(X, I, w) \in \mathcal{E}_{k,p}$, $k < p$, and $m_0 < n_0$. Let $I' = I \cap]m_0, n_0]$ and $A \in \mathcal{A}_{m_0}$. Then if $X' = (\pi_A^{-1}(X))_{n_0}$ we have $(X', I', w') \in \mathcal{E}_{k,p}$ where $w' = w(\text{card} I / \text{card} I')^{\alpha(k)}$.*

Proof. It suffices to prove that X' is $(I', \varphi_{k+1,p})$ -thin. Consider $m, n \in I'$, $m < n$, so that $m_0 < m < n \leq n_0$. Consider $A_1 \in \mathcal{A}_m$, and set $A_2 = \pi_A(A_1) \subset A$, so

that $A_2 \in \mathcal{A}_m$. Since X is $(m, n, \varphi_{k+1,p})$ -thin, there exists $C \subset A_2$, $C \in \mathcal{B}_n$, with $C \cap X = \emptyset$, $\varphi_{k+1,p}(\pi_{A_2}^{-1}(C)) \geq 1$. Let $C' = A_1 \cap \pi_{A_2}^{-1}(C)$, so that $C' \in \mathcal{B}_n$.

We observe that if a set B does not depend on the coordinates of rank $\leq m$, we have

$$\pi_{A_1}^{-1}(B) = B = \pi_{A_1}^{-1}(B \cap A_1).$$

Using this for $B = \pi_{A_2}^{-1}(C)$, we get that $\pi_{A_1}^{-1}(C') = \pi_{A_2}^{-1}(C)$, and consequently $\varphi_{k+1,p}(\pi_{A_1}^{-1}(C')) \geq 1$.

It remains only to prove that $C' \cap X' = \emptyset$. This is because on A_1 the maps π_A and π_{A_2} coincide, so that, since $C' \subset A_1$, we have $\pi_A(C') = \pi_{A_2}(C') \subset C$ and hence $\pi_A(C') \cap X = \emptyset$. Thus $C' \cap \pi_A^{-1}(X) = \emptyset$ and since $C' \in \mathcal{B}_n$ we have $C' \cap X' = \emptyset$. \square

Given p , the proof of Theorem 5.1 will go by decreasing induction over k . For $k = p$, the result is true since by Proposition 3.2 we have $\varphi_{p,p}(T) = \psi(T) \geq 2^5 \geq c_k$.

Now we proceed to the induction step from $q+1$ to q . Considering $F \subset \mathcal{C}_{q,p}$, with $w(F) < c_q$, our goal is to show that $\cup F \neq T$. Since $\mathcal{C}_{q,p} = \mathcal{C}_{q+1,p} \cup \mathcal{E}_{q,p}$ we have $F = F_1 \cup F_2$, $F_1 \subset \mathcal{E}_{q,p}$, $F_2 \subset \mathcal{C}_{q+1,p}$.

Let $F'_2 = F_2 \cap \bigcup_{k < q} \mathcal{D}_k$. When $(X, I, w) \in \mathcal{D}_k$ we have $w \geq 2^{-k} \geq 2^{-q}$, and thus

$$2^{-q} \text{card} F'_2 \leq w(F'_2) \leq w(F) \leq c_q \leq 2^5$$

so that $\text{card} F'_2 \leq 2^{q+5}$. Also, for $(X, I, w) \in \mathcal{D}_k$ we have $\text{card} I \leq N(k)$, so that if we set

$$I^* = \bigcup \{I; (X, I, w) \in F'_2\} \quad (5.4)$$

we have

$$\text{card} I^* \leq t' := 2^{q+5} N(q-1). \quad (5.5)$$

When $(X, I, w) \in \mathcal{E}_{q,p}$ we have $w \geq 2^{-q}$. Thus

$$2^{-q} \text{card} F_1 \leq w(F_1) \leq w(F) \leq c_q \leq 2^5$$

and thus $s := \text{card} F_1 \leq 2^{q+5}$. Also, when $(X, I, w) \in \mathcal{E}_{q,p}$ we have

$$2^{-q} \left(\frac{M(q)}{\text{card} I} \right)^{\alpha(q)} = w \leq 2^5$$

so that

$$\text{card} I \geq M(q) 2^{-(q+5)/\alpha(q)} \quad (5.6)$$

and hence, if

$$t = 2^{q+8} + t' \quad (5.7)$$

under (5.2) we have $\text{card} I \geq st$ where $s = \text{card} F_1$. We follow the proof of Proposition 3.6. We appeal to Roberts's selection lemma to enumerate F_1 as $(X_\ell, I_\ell, w_\ell)_{\ell \leq s}$ and find sets $J_1 \prec J_2 \prec \dots \prec J_s$ with $\text{card} J_\ell = t$ and $J_\ell \subset I_\ell$. We then appeal to the counting argument of Proposition 3.6, but instead of allowing in (3.17) all the values of $u(\ell) \leq t - 1$, we now restrict the choice of $u(\ell)$ by

$$u(\ell) \in U_\ell = \{u; 1 \leq u \leq t - 1, I^* \cap]i_{u,\ell}, i_{u+1,\ell}] = \emptyset\}.$$

We observe that by (5.5) and (5.7) we have $\text{card} U_\ell \geq 2^{q+8} - 1$.

The counting argument then allows us to find \vec{u} such that (since $w(F_2) \leq 2^5$)

$$S(\vec{u}) \leq \frac{2}{2^{q+8} - 1} w(F_2) \leq 2^{-q-1}.$$

Using the notation (3.18) we have thus constructed intervals $W_\ell =]m_\ell, n_\ell]$, $\ell \leq s$, with $n_\ell \leq m_{\ell+1}$, in such a manner that X_ℓ is $(m_\ell, n_\ell, \varphi_{q+1,p})$ -thin and that if F_3 is defined by (3.19) we have that

$$w(F_3) \leq 2^{-q-1} \leq \frac{1}{4}. \quad (5.8)$$

Moreover, if $W = \bigcup_{\ell \leq s}]m_\ell, n_\ell]$ we have ensured that

$$(X, I, w) \in F'_2 \implies W \cap I = \emptyset,$$

so that in particular if we define F_4 by (3.20) we have

$$(X, I, w) \in F_4, (X, I, w) \in \bigcup_{k < q} \mathcal{D}_k \implies W \cap I = \emptyset. \quad (5.9)$$

As before, (5.8) implies that if $(X, I, w) \in \mathcal{D}_k \cap F_3$, then $k \geq q$. Let us define the classes F_3^ℓ , $\ell \leq s$ by

$$F_3^\ell = \{(X, I, w) \in F_3; \text{card}(I \cap W_\ell) \geq 2^{-q-6} \text{card} I\},$$

so that, since $s \leq 2^{q+5}$, we have $F_3 = \bigcup_{\ell \leq s} F_3^\ell$.

Lemma 5.3 *Consider $(X, I, w) \in F_3^\ell$ and $A \in \mathcal{A}_{m_\ell}$. Then we can find (X', I', w') in $\mathcal{C}_{q+1,p}$ with $X' \supset \pi_A^{-1}(X)$, $X' \in \mathcal{B}_{n_\ell}$, $w' \leq 2w$.*

Proof. If $(X, I, w) \in \mathcal{D}$ we have already proved this statement in the course of the proof of Proposition 3.5, so, since $\mathcal{C}_{q+1,p} = \mathcal{D} \cup \bigcup_{q+1 \leq r \leq p} \mathcal{E}_{r,p}$, it suffices to consider the case where $(X, I, w) \in \mathcal{E}_{r,p}$, $r \geq q+1$. In that case, if $I' = I \cap W_\ell$, we have

$$\left(\frac{\text{card} I}{\text{card} I'} \right)^{\alpha(r)} \leq (2^{q+6})^{\alpha(r)} \leq 2$$

and the result follows from Lemma 5.2. \square

Corollary 5.4 *Consider $A \in \mathcal{A}_{m_\ell}$. Then there is $A' \in \mathcal{A}_{n_\ell}$ such that $A' \subset A$, $A' \cap X_\ell = \emptyset$ and $A' \cap \cup F_3^\ell = \emptyset$.*

Proof. Lemma 5.3 shows that $\pi_A^{-1}(\cup F_3^\ell) \subset C'$, where $C' \in \mathcal{B}_{n_\ell}$ and $\varphi_{q+1,p}(C') \leq 2w(F_3^\ell) \leq 1/2$. Since X_ℓ is $(m_\ell, n_\ell, \varphi_{q+1,p})$ -thin, we can find $C \in \mathcal{B}_{n_\ell}$, $C \subset A$, $C \cap X = \emptyset$ with $\varphi_{q+1,p}(\pi_A^{-1}(C)) \geq 1$. Thus we cannot have $\pi_A^{-1}(C) \subset C'$ and hence since both these sets belong to \mathcal{B}_{n_ℓ} we can find $A_1 \in \mathcal{A}_{n_\ell}$ with

$$A_1 \subset \pi_A^{-1}(C) \setminus C' \subset \pi_A^{-1}(C) \setminus \pi_A^{-1}(\cup F_3^\ell).$$

Thus $A' = \pi_A(A_1) \in \mathcal{A}_{n_\ell}$, $A' \cap \cup F_3^\ell = \emptyset$, $A' \subset C$, so that $A' \cap X_\ell = \emptyset$. \square

We now construct a map $\Xi : T \rightarrow T$ with the following properties. For $\vec{y} \in T$, $\vec{z} = \Xi(\vec{y})$ is such that $\vec{z}_i = \vec{y}_i$ whenever $i \notin W = \bigcup_{\ell \leq s} [m_\ell, n_\ell]$. Moreover, for each ℓ , and each $A \in \mathcal{A}_{m_\ell}$, there exists $A' \in \mathcal{A}_{n_\ell}$ with

$$\vec{y} \in A \implies \Xi(\vec{y}) \in A',$$

and A' satisfies $A' \cap X_\ell = \emptyset$ and $A' \cap \cup F_3^\ell = \emptyset$.

The existence of this map is obvious from Corollary 5.4. It satisfies

$$\ell \leq s \implies \Xi(T) \cap X_\ell = \emptyset, \quad \Xi(T) \cap \cup F_3^\ell = \emptyset. \quad (5.10)$$

It has the further property that for each integer j the first j coordinates of $\Xi(\vec{y})$ depend only on the first j coordinates of \vec{y} .

We recall that F_4 is as in (3.20).

Lemma 5.5 *We have $\varphi_{q+1,p}(\Xi^{-1}(\cup F_4)) < c_{q+1}$.*

Proof of Theorem 5.1. Using the induction hypothesis $\varphi_{q+1,p}(T) \geq c_{q+1}$ we see that there is \vec{y} in $T \setminus \Xi^{-1}(\cup F_4)$, so that $\Xi(\vec{y}) \notin \cup F_4$. Combining with (5.10) we see that $\Xi(\vec{y}) \notin \bigcup_{\ell \leq s} X_\ell = \cup F_1$, $\Xi(\vec{y}) \notin \cup F_3$, so that $\Xi(\vec{y}) \notin \cup F$. \square

Proof of Lemma 5.5. We prove that if $(X, I, w) \in F_4$, then $\varphi_{q+1,p}(\Xi^{-1}(X)) \leq w^{2^{2\alpha(q)}}$. This suffices since $w(F_4) < c_q$.

Case 1. $(X, I, w) \in \mathcal{D}_k$, $k < q$.

In that case, by (5.9) we have $I \cap W = \emptyset$, so that $\Xi^{-1}(X) = X$ and thus $\varphi_{q+1,p}(\Xi^{-1}(X)) = \varphi_{q+1,p}(X) \leq w$.

Case 2. We have $(X, I, w) \in \mathcal{D}_k$, $k \geq q$.

We use Lemma 3.1 with $J = \mathbb{N}^* \setminus W$ and the fact that $\alpha(k) \leq \alpha(q) \leq (q+5)^{-3}$. This has already been done in the previous section.

Case 3. $(X, I, w) \in \mathcal{E}_{r,p}$ for some $q+1 \leq r < p$.

In a first stage we prove the following. Whenever $m, n \in I$ are such that $m < n$, and $]m, n] \cap W = \emptyset$, then $\Xi^{-1}(X)$ is $(m, n, \varphi_{r+1,p})$ -thin. Since for each integer j the first j coordinates of $\Xi(\vec{y})$ depend only on the first j coordinates of \vec{y} , whenever $A \in \mathcal{A}_m$ there is $A' \in \mathcal{A}_m$ with $\Xi(A) \subset A'$. Since X is $(m, n, \varphi_{r+1,p})$ -thin we can find $C' \in \mathcal{B}_n$ with $C' \cap X = \emptyset$, $C' \subset A'$, and $\varphi_{r+1,p}(\pi_{A'}^{-1}(C')) \geq 1$. Let $C = \Xi^{-1}(C') \in \mathcal{B}_n$. We observe that $C \cap \Xi^{-1}(X) = \emptyset$ and we now prove that

$$\Xi(\pi_A(\pi_{A'}^{-1}(C'))) \subset C'. \quad (5.11)$$

Consider τ_1, \dots, τ_m and τ'_1, \dots, τ'_m such that

$$\begin{aligned} A &= \{\vec{z} \in T; \forall i \leq m, z_i = \tau_i\} \\ A' &= \{\vec{z} \in T; \forall i \leq m, z_i = \tau'_i\}. \end{aligned}$$

Consider $\vec{y} \in \pi_{A'}^{-1}(C')$. Then there exists $\vec{y}' \in C'$ with $y_i = y'_i$ for $i > m$. Thus $\vec{y}'' = \pi_A(\vec{y})$ is such that $y''_i = \tau_i$ for $i \leq m$, and $y''_i = y'_i$ for $i > m$, so that $\vec{z} = \Xi(\vec{y}'')$ is such that $z_i = \tau'_i$ for $i < m$. Moreover $z_i = y''_i$ for $i \notin W$, and since $]m, n] \cap W = \emptyset$, we have $z_i = y''_i = y'_i$ for $m < i \leq n$. Since $C' \subset A'$, we have $y'_i = \tau'_i$ for $i < m$, so that $z_i = y'_i$ for all $i \leq n$, and thus $\vec{z} \in C'$ since $\vec{y}' \in C' \in \mathcal{B}_n$. Since \vec{y} is arbitrary this proves (5.11), which implies that

$$\pi_{A'}^{-1}(C') \subset \pi_A^{-1}(\Xi^{-1}(C')) = \pi_A^{-1}(C),$$

so that $\varphi_{r+1,p}(\pi_A^{-1}(C)) \geq 1$ and we have proved that $\Xi^{-1}(X)$ is $(m, n, \varphi_{r+1,p})$ -thin.

For each $\ell \leq 1$, consider the largest element $i(\ell)$ of I that is $\leq m_\ell$. (Trivial modifications of the argument take care of the case where I has no elements $\leq m_\ell$). Let

$$I' = I \setminus (W \cup \{i(1), \dots, i(s)\}),$$

so that, since $\text{card}(I \setminus W) \geq \text{card} I / 2$, we have

$$\text{card} I' \geq \frac{\text{card} I}{2} - s \geq \frac{\text{card} I}{2} - 2^{q+5} \geq \frac{\text{card} I}{4},$$

using (5.6) and (5.2). We claim that $\Xi^{-1}(X)$ is $(m, n, \varphi_{r+1,p})$ -thin whenever $m < n$, $m, n \in I'$. To see this, consider the smallest element n' of I such that $m < n'$. Then

$n' \leq n$, so it suffices to show that $\Xi^{-1}(X)$ is $(m, n', \varphi_{r+1,p})$ -thin. By the first part of the proof, it suffices to show that $W \cap]m, n'] = \emptyset$. Assuming $W_\ell \cap]m, n'] \neq \emptyset$, we see that $m_\ell < n'$. Since $m \notin W_\ell$ we have $m \leq i(\ell)$ and since $m \neq i(\ell)$, we have $m < i(\ell) \leq m_\ell$, contradicting the choice of n' .

Let $w' = w(\text{card} I / \text{card} I')^{\alpha(q)} \leq w 2^{2\alpha(q)}$. It should then be obvious that $(\Xi^{-1}(X), I', w') \in \mathcal{E}_{r,q}$, so that $\varphi_{q+1,p}(\Xi^{-1}(X)) \leq w 2^{2\alpha(q)}$. \square

6 Exhaustivity

Lemma 6.1 *Consider $B \in \mathcal{B}$ and $a > 0$. If $\nu_k(B) < a$ then*

$$\{p; \varphi_{k,p}(B) < a\} \in \mathcal{U}.$$

Proof. By definition of $\nu_k = \varphi_{\mathcal{C}_k}$, there exists a finite set $F \subset \mathcal{C}_k = \mathcal{D} \cup \bigcup_{r \geq k} \mathcal{E}_r$ with $w(F) < a$ and $\cup F \supset B$. By definition of \mathcal{E}_r , for $(X, I, w) \in \mathcal{E}_r$ we have

$$\{p; (X, I, w) \in \mathcal{E}_{r,p}\} \in \mathcal{U},$$

so that since $\mathcal{C}_{k,p} = \mathcal{D} \cup \bigcup_{k \leq r < p} \mathcal{E}_{r,p}$ we have $\{p; F \subset \mathcal{C}_{k,p}\} \in \mathcal{U}$ and thus $\varphi_{k,p}(B) \leq w(F) < a$ for these p . \square

Corollary 6.2 *We have $\nu(T) \geq 16$.*

Proof. By Lemma 6.1, and since $\varphi_{1,p}(T) \geq c_1 = 16$ by Theorem 5.1.

The next lemma is a kind of converse to Lemma 6.1, that lies much deeper.

Lemma 6.3 *Consider $B \in \mathcal{B}$ with $\nu_k(B) \geq 4$. Then*

$$\{p; \varphi_{k,p}(B) \geq 1\} \in \mathcal{U}.$$

Proof. Consider n such that $B \in \mathcal{B}_n$, and assume for contradiction that

$$U = \{p; \varphi_{k,p}(B) < 1\} \in \mathcal{U}.$$

Thus, for $p \in U$, we can find $F_p \subset \mathcal{C}_{k,p}$ with $B \subset \cup F_p$ and $w(F_p) \leq 1$. Let

$$\begin{aligned} F_p^1 &= \{(X, I, w) \in F_p; \text{card}(I \cap \{1, \dots, n\}) \geq \text{card} I / 2\} \\ F_p^2 &= F_p \setminus F_p^1 = \{(X, I, w) \in F_p; \text{card}(I \cap \{1, \dots, n\}) < \text{card} I / 2\}. \end{aligned}$$

Using Lemmas 3.1 and 5.2 we find a family F_p^\sim of triples (X', I', w') in $\mathcal{C}_{k,p}$ with $\cup F_p^\sim \supset \cup F_p^1$, $w(F_p^\sim) \leq 2$ and $I' \subset \{1, \dots, n\}$, $X' \in \mathcal{B}_n$, so that $\cup F_p^\sim \in \mathcal{B}_n$.

We claim that $B \subset \cup F_p^\sim$. For, otherwise, since B and $\cup F_p^\sim$ both belong to \mathcal{B}_n , we can find $A \in \mathcal{A}_n$ with $A \subset B \setminus \cup F_p^\sim$, so that $A \subset \cup F_p^2$. By Lemma 5.2 again (or, to be exact, its obvious extension to the case $n_0 = \infty$) and Lemma 3.1 we get

$$\varphi_{k,p}(T) = \varphi_{k,p}(\pi_A^{-1}(\cup F_p^2)) \leq 2w(F_p^2) \leq 2,$$

which is impossible because $\varphi_{k,p}(T) \geq 16$.

Using (3.12) and (4.3) we see that there exists a finite collection \mathcal{G} of triplets (X, I, w) such that $F_p^\sim \subset \mathcal{G}$ for all p . Thus there exists a set F such that $\{p \in U; F_p^\sim = F\} \in \mathcal{U}$. It follows from (4.4) that $F \subset \mathcal{C}_k$ and it is obvious that $B \subset \cup F$ and $w(F) \leq 2$, so that $\nu_k(B) \leq 2$, a contradiction. \square

Corollary 6.4 *Consider a triplet (X, I, w) and k with $\text{card} I \leq M(k)$ and*

$$w = 2^{-k} \left(\frac{M(k)}{\text{card} I} \right)^{\alpha(k)}.$$

Assume that X is $(I, \nu_{k+1}/4)$ -thin, i.e.

$$\forall m, n \in I, \quad m < n, \quad \forall A \in \mathcal{A}_m, \quad \exists C \in \mathcal{B}_n, \quad C \cap X = \emptyset, \quad \nu_{k+1}(\pi_A^{-1}(C)) \geq 4. \quad (6.1)$$

Then $(X, I, w) \in \mathcal{E}_k$.

Proof. If $\nu_{k+1}(\pi_A^{-1}(C)) \geq 4$ then by Lemma 6.3 we have $\{p; \varphi_{k+1,p}(\pi_A^{-1}(C)) \geq 1\} \in \mathcal{U}$ and

$$\{p; (X, I, w) \in \mathcal{E}_{k,p}\} \supset \bigcap \{p; \varphi_{k+1,p}(\pi_A^{-1}(C)) \geq 1\} \in \mathcal{U},$$

where the intersection is over all sets A, C as in (6.1). \square

Lemma 6.5 *Consider a sequence (E_i) of \mathcal{B} , and assume that these sets do not depend on the coordinates of rank $\leq m$ for a certain m . Assume that*

$$\forall n, \quad \nu_k \left(\bigcup_{i \leq n} E_i \right) < 4.$$

Then for each $\alpha > 0$ there is $C \in \mathcal{B}$, that does not depend on the coordinates of rank $\leq m$, and such that $\nu_k(C) \leq 8$ and $\nu_k(E_i \setminus C) \leq \alpha$ for each i .

Proof. For each n , let

$$U_n = \left\{ p; \varphi_{k,p} \left(\bigcup_{i \leq n} E_i \right) < 4 \right\}$$

so that $U_n \in \mathcal{U}$ by Lemma 6.1. For $p \in U_n$ we can find $F_{n,p} \subset \mathcal{C}_{k,p}$ with $\bigcup_{i \leq n} E_i \subset \bigcup F_{n,p}$ and $w(F_{n,p}) \leq 4$. For $r \geq m+1$ we define

$$F_{n,p}^r = \left\{ (X, I, w) \in F_{n,p}; \text{card}(I \cap \{m+1, \dots, r-1\}) \leq \frac{1}{2} \text{card} I; \right. \\ \left. \text{card}(I \cap \{m+1, \dots, r\}) \geq \frac{1}{2} \text{card} I \right\},$$

and we define

$$F'_{n,p} = \left\{ (X, I, w) \in F_{n,p}; \text{card}(I \cap \{1, \dots, m\}) \geq \frac{1}{4} \text{card} I \right\}.$$

We use Lemmas 3.1 and 5.2 to find a set $B \in \mathcal{B}_m$ with $\varphi_{k,p}(B) \leq 8$ and $B \supset \bigcup F'_{n,p}$ so that since $\varphi_{k,p}(T) \geq 16$ we have $B \neq T$ and thus there exists $A_{n,p} \in \mathcal{A}_m$ with $A_{n,p} \cap \bigcup F'_{n,p} = \emptyset$. We use again Lemmas 3.1 and 5.2 to see that for $(X, I, w) \in F_{n,p}^r$ we can find $w' \leq 2w$ such that $(X', I', w') = ((\pi_{A_{n,p}}^{-1}(X))_r, I \cap \{m+1, \dots, r\}, w') \in \mathcal{C}_{k,p}$. We observe that X' does not depend on the coordinates of rank $\leq m$. Let $F_{n,p}^r$ be the collection of the sets (X', I', w') for $(X, I, w) \in F_{n,p}^r$ so that $w(F_{n,p}^r) \leq 2w(F_{n,p}^r)$. We claim that if $E_i \in \mathcal{B}_j$ we have

$$E_i \subset \bigcup_{r \leq j} \bigcup F_{n,p}^r. \quad (6.2)$$

Otherwise, since both sets depend only on the coordinates of rank $\geq m$ and $\leq j$, and since $A_{n,p} \in \mathcal{A}_m$, we would find $A \in \mathcal{A}_j$ with $A \subset A_{n,p}$ and $A \subset E_i \setminus \bigcup_{r \leq j} \bigcup F_{n,p}^r$. Since $\pi_{A_{n,p}}^{-1}(X) \cap A_{n,p} \supset X \cap A_{n,p}$, this shows that $A \subset E_i \setminus \bigcup_{r \leq j} \bigcup F_{n,p}^r$. Since $A_{n,p} \cap \bigcup F_{n,p}^r = \emptyset$ for $r \leq j$, and since $E_i \subset \bigcup F_{n,p}$, we have $A \subset \bigcup F_{n,p}''$, where

$$F_{n,p}'' = F_{n,p} \setminus \left(F'_{n,p} \cup \bigcup_{r \leq j} F_{n,p}^r \right) \subset \left\{ (X, I, w) \in F_{n,p}; \text{card} I \cap \{j+1, \dots\} \geq \frac{1}{4} \text{card} I \right\}.$$

A new application of Lemmas 3.1 and 5.2 then shows that $T = \pi_A^{-1}(A)$ satisfies $\varphi_{k,p}(T) \leq 8$, and this is impossible. So we have proved (6.2).

Given r , we prove using (3.12) and (4.3) that $F_{n,p}^r \subset \mathcal{G}^r$ where \mathcal{G}^r is finite and does not depend on n or p . It should then be clear using (4.1) how to take limits as

$p \rightarrow \infty$, $n \rightarrow \infty$ to define for $r \geq m+1$ sets $F^r \subset \mathcal{C}_k$ with $\sum_{r \geq m+1} w(F^r) \leq 8$ such that $E_i \subset \bigcup_{r \leq j} \cup F^r$ provided $E_i \in B_j$. The elements of F^r are of the type (X, I, w) where X does not depend on the coordinates of rank $\leq m$, and $X \in \mathcal{B}_r$.

Consider r_0 such that $\sum_{r > r_0} w(F^r) < \alpha$ and let $C = \bigcup_{r \leq r_0} \cup F^r$. Then $\nu_k(C) \leq \sum_{r \leq r_0} w(F^r) \leq 8$ and

$$E_i \setminus C \subset \bigcup_{r_0 \leq r \leq j} \cup F^r$$

so that $\nu_k(E_i \setminus C) \leq \alpha$. □

Lemma 6.6 *Consider $k > 0$, $\alpha > 0$, $B \in \mathcal{B}_m$, a disjoint sequence (E_i) of \mathcal{B} . Then we can find $n > m$, a set $B' \in \mathcal{B}_n$, $B' \subset B$ such that B' is $(m, n, \nu_{k+1}/4)$ -thin and*

$$\limsup_{i \rightarrow \infty} \nu_k((B \cap E_i) \setminus B') \leq \alpha.$$

Proof. Nearly identical to that of Lemma 3.8, using Lemma 6.5, and since $\nu_k(T) \geq 16$.

Proof that ν is exhaustive. For each k we show that ν is 2^{-k} exhaustive following the method of Proposition 3.5, and using that by Corollary 6.4, if X is $(I, \nu_{k+1}/4)$ -thin where $\text{card} I = M(k)$, then $(X, I, 2^{-k}) \in \mathcal{E}_k$, so that $\nu(X) \leq \nu_k(X) \leq 2^{-k}$. □

7 Proof of Theorems 1.2 to 1.4

The simple arguments we present here are essentially copied from the paper of Roberts [R], and are provided for the convenience of the reader.

To prove Theorem 1.4, we simply consider the space \mathcal{L}_0 of real-valued functions defined on the Cantor set that are \mathcal{B} -measurable, provided with the topology induced by the distance d such that

$$d(f, 0) = \sup\{\varepsilon; \nu(\{|f| \geq \varepsilon\}) \geq \varepsilon\}, \tag{7.1}$$

where ν is the submeasure of Theorem 1.1. We consider the \mathcal{L}_0 -valued vector measure θ given by $\theta(A) = 1_A$. Thus $d(0, \theta(A)) = \nu(A)$, which makes it obvious that θ is exhaustive and does not have a control measure. Let us also note that d satisfies the nice formula

$$d(f + g, 0) \leq d(f, 0) + d(g, 0),$$

as follows from the relation $\{|f + g| \geq \varepsilon_1 + \varepsilon_2\} \subset \{|f| \geq \varepsilon_1\} \cup \{|g| \geq \varepsilon_2\}$.

We start the proof of Theorem 1.2. We first observe that the submeasure ν of Theorem 1.1 is strictly positive, i.e., $\nu(A) > 0$ if $A \neq \emptyset$. This follows from subadditivity and the fact that by construction we have $\nu(A) = \nu(A')$ for $A, A' \in \mathcal{A}_n$ and any n .

We consider the distance d on \mathcal{B} given by

$$d(A, B) = \nu(A \Delta B),$$

where Δ denotes the symmetric difference. It is simple to see that the completion $\widehat{\mathcal{B}}$ of \mathcal{B} with respect to this distance is still a Boolean algebra, the operations being defined by continuity, and that ν extends to $\widehat{\mathcal{B}}$ in a positive submeasure, still denoted by ν . We claim that ν is exhaustive. To see this, consider a disjoint sequence (E_n) in $\widehat{\mathcal{B}}$. Consider $\varepsilon > 0$, and for each n find A_n in \mathcal{B} with $\nu(A_n \Delta E_n) \leq \varepsilon 2^{-n}$. Let $B_n = A_n \setminus (A_1 \cup \dots \cup A_{n-1})$, so that, since $E_n = E_n \setminus (E_1 \cup \dots \cup E_{n-1})$ we have

$$\nu(B_n \Delta E_n) \leq \sum_{m \leq n} \nu(E_m \Delta A_m) \leq \sum_{m \leq n} \varepsilon 2^{-m} \leq \varepsilon. \quad (7.2)$$

Since the sequence (B_n) is disjoint in \mathcal{B} , we have $\lim_{n \rightarrow \infty} \nu(B_n) = 0$, and by (7.2) we have $\limsup_{n \rightarrow \infty} \nu(E_n) \leq \varepsilon$. As ε is arbitrary, this proves the result.

Consider now a decreasing sequence (A_n) of $\widehat{\mathcal{B}}$. The fundamental observation is that it is a Cauchy sequence for d . Otherwise, we could find $\varepsilon > 0$ and numbers $m(k) < n(k) \leq m(k+1) < n(k+1) \dots$ with $\nu(A_{n(k)} \setminus A_{m(k)}) \geq \varepsilon$, and this contradicts exhaustivity.

The limit of the sequence (A_n) in $\widehat{\mathcal{B}}$ is clearly the infimum of this sequence. This shows that $\widehat{\mathcal{B}}$ is σ -complete and that ν is continuous.

It follows that ν is countably subadditive, i.e.

$$\nu\left(\bigcup_{n \geq 1} A_n\right) \leq \sum_{n \geq 1} \nu(A_n). \quad (7.3)$$

This is because for each m , if we set $B_m = \bigcup_{n \geq 1} A_n \setminus (\bigcup_{1 \leq n \leq m} A_n)$ we have

$$\nu\left(\bigcup_{n \geq 1} A_n\right) \leq \nu\left(B_m \cup \bigcup_{1 \leq n \leq m} A_n\right) \leq \nu(B_m) + \sum_{1 \leq n \leq m} \nu(A_n)$$

and that $\nu(B_m) \rightarrow 0$ since ν is continuous, since the sequence (B_n) decreases and since $\bigcap_{m \geq 1} B_m = 0$ (the smallest element of $\widehat{\mathcal{B}}$).

Lemma 7.1 *Consider $A \in \mathcal{B}$, and countable collections \mathcal{C}_n , $n \geq 1$ such that $A \subset \bigcup \mathcal{C}_n$ for each n . Then for each $\eta > 0$ there is $A' \subset A$ with $\nu(A \setminus A') \leq \eta$ such that for each n , A is covered by a finite subset of \mathcal{C}_n .*

Proof. Enumerate \mathcal{C}_n as $(C_{n,m})_{m \geq 1}$. Since $A \subset \cup \mathcal{C}_n$, we have $\bigcap_k (A \setminus \bigcup_{m \leq k} C_{n,m}) = 0$, so that by continuity of ν there exists $k(n)$ with $\nu(A \setminus \bigcup_{m \leq k(n)} C_{n,k}) \leq \eta 2^{-n}$. The set $A' = \bigcap_n \bigcup_{m \leq k(n)} C_{n,k}$ is for each n covered by a finite subset of \mathcal{C}_n and it satisfies $\nu(A \setminus A') \leq \eta$ by (7.3). \square

Consider a measure μ on $\widehat{\mathcal{B}}$. Then μ is not absolutely continuous with respect to ν on \mathcal{B} , so that we can find $\varepsilon > 0$ and for each n a set $B_n \in \mathcal{B}$ with $\nu(B_n) \leq 2^{-n}$ and $\mu(B_n) \geq \varepsilon$. Let $A_n = \bigcup_{m \geq n} B_m$. By (7.3) we have $\nu(A_n) \leq \sum_{m \geq n} 2^{-m} \leq 2^{-n+1}$ so that if $A = \bigcap_{n \geq 1} A_n$ we have $\nu(A) = 0$ and thus $A = 0$. But by monotonicity we have $\mu(A_n) \geq \varepsilon$, so that μ is not continuous.

On the other hand, ν is not absolutely continuous with respect to μ on \mathcal{B} , so for some $\varepsilon > 0$ and each n we can find $B_n \in \mathcal{B}$ with $\nu(B_n) \geq \varepsilon$ and $\mu(B_n) \leq 2^{-n}$. Let $A_n = \bigcup_{m \geq n} B_m$ and $A = \bigcap_{n \geq 1} A_n$, so that $\nu(A_n) \geq \varepsilon$ and $\nu(A) \geq \varepsilon$ by continuity of ν . We use Lemma 7.1 with $\eta = \varepsilon/2$, $\mathcal{C}_n = \{B_m; m \geq n\}$, $A = \bigcap_{n \geq 1} A_n$. We have $\nu(A') \geq \varepsilon/2$ since $\nu(A) \geq \varepsilon$, so that $A' \neq 0$. For each n , since μ is subadditive, and since A' can be covered by a finite subset of \mathcal{C}_n we have $\mu(A') \leq \sum_{m \geq n} 2^{-m} = 2^{-n+1}$. Thus $\mu(A') = 0$, and μ hence is not positive. This concludes the proof of Theorem 1.2.

To prove Theorem 1.3, we first observe that $\widehat{\mathcal{B}}$ satisfies the countable chain condition, since ν is positive and exhaustive. We prove that it also satisfies the general distributive law. Given a sequence (Π_n) of partitions and $m \in \mathbb{N}^*$, Lemma 7.1 produces a set C_m with $\nu(C_m^c) \leq 2^{-m}$ such that C_m is finitely covered by every partition Π_n . And $C_1, C_2 \setminus C_1, C_3 \setminus (C_1 \cup C_2), \dots$ is the required partition. This concludes the proof of Theorem 1.3

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